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ON THE PRANDTL INTEGRO-DIFFERENTIAL EQUATION

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1. Let us consider the wing of finite span equal to 2a located symmetrically with respect to the yz-plane. We will assume that the z-axis direction coincides with the direction of the air flow at infinity and that the x-axis is directed perpendicularly to the plane symmetric to the wing. Let $b(x)$ be the wing chord profile of the corresponding x abscissa and $\Gamma(x)$ the circulation along this profile. Let us denote by $\alpha(x)$ the geometric angle of attack and by V the air velocity at infinity.

On the basis of wing symmetry we have

$$\Gamma(x) = \Gamma(-x) \; ; \quad b(x) = b(-x); \quad \alpha(x) = \alpha(-x); \quad (-a \leq x \leq a) \tag{1.1}$$

In wing theory it is proved that $\Gamma(x)$ satisfies the following integro-differential equation due to Prandtl (see [1], p.194; [3])

$$\frac{8\pi}{m\, b(x)} \Gamma(x) - \int_{-a}^{a} \frac{\Gamma'(t)}{t - x}\, dt = 4\pi V \alpha(x) \tag{1.2}$$

where m is a constant which usually equals 2π; a more accurate value of this number is 5.5 (see [1], p. 194).

Equation (1.2) is singular since it contains an integral which must be interpreted in the sense of the Cauchy principal value. Consequently it is impossible to apply the usual integral equation theory to it. We show, below, that equation (1.2) may be replaced by a Fredholm equation having a rather simple structure, so that it may be effectively solved in many important practical cases (see examples below).

2. We limit ourselves to the consideration of the case when

$$b(x) = \frac{\sqrt{a^2 - x^2}}{p(x)} \qquad (-a \leq x \leq a) \qquad (2.1)$$

where $p(x)$ is an analytic function on the segment $[-a,a]$ which satisfies the condition

$$p(x) > 0, \quad p(x) = p(-x)\ , \qquad (-a \leq x \leq a) \qquad (2.2)$$

Evidently, then, there exists a simply connected region T containing a line within the segment $[-a,a]$ and bounded by a smooth closed curve L such that the function $p(\zeta)$ is holomorphic in $T + L$. Cutting the point of the segment $[-a, a]$ out of T we obtain the doubly connected region which we denote by T^*. The function

$$b(\zeta) = \frac{\sqrt{a^2 - \zeta^2}}{p(\zeta)} \qquad (2.3)$$

evidently will be holomorphic in T^* while we have in view below that branch of the function which satisfies the condition [1]

$$b_+(x) = - b_-(x) = b(x) > 0 \qquad (-a \leq x \leq a) \qquad (2.4)$$

3. Let us consider the Cauchy type integral

$$\phi(\zeta) = \frac{1}{2\pi i} \int_{-a}^{a} \frac{\Gamma(t)}{t - \zeta}\, dt \qquad (3.1)$$

which evidently represents a function holomorphic everywhere on the plane except the segment $[-a,a]$.

In what follows, we assume that $\Gamma(x)$ is a function, continuous in the Holder sense, on $[-a,a]$ and that its derivative is

$$\Gamma'(x) = \frac{\Gamma^*(x)}{(a^2 - x^2)^\delta} \qquad \delta < 1 \qquad (3.2)$$

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1. The symbols $f_+(x)$ and $f_-(x)$ generally denote the limiting value of $f(\zeta)$ in the neighborhood of $[-a,a]$ when $\zeta \to x$ of this segment from the upper or lower half-plane respectively.

where $\Gamma^*(x)$ is a function continuous in the Holder sense on $[-a,a]$, We assume that $\alpha(x)$ is also continuous in the Holder sense on $[-a,a]$ In practice α is usually constant.

From (3.1) by differentiating with respect to ζ and integrating by parts, taking into account (1.1), we obtain

$$\phi'(\zeta) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{\Gamma'(t)\,dt}{t-\zeta} - \frac{a\Gamma(a)}{\pi i(a^2-\zeta^2)} \tag{3.3}$$

In [2] N. Muskhelishvili established that $\phi(\zeta)$ and $\phi'(\zeta)$ for the above assumptions with respect to $\Gamma(x)$, are continuously prolonged from the interior point of the segment $[-a,a]$ up to the sides of upper half-plane as well as the lower. On the end-points of the segment $[-a,a]$ $\phi(\zeta)$ may have a singularity only of logarithmic type and the integral term on the right side of (3.3) may have singularities at these points of order less than 1 .

On the basis of the wellknown properties of the Cauchy type integrals, from (3.1) and (3.3) by passing to the limit, we obtain

$$\Gamma(x) = \phi_+(x) - \phi_-(x) \tag{3.4}$$

$$\int_{-a}^{a} \frac{\Gamma'(t)}{t-x}\,dt = \pi i\left[\phi_+(x) + \phi_-'(x)\right] + \frac{2a\Gamma(a)}{a^2-x^2} \tag{3.5}$$

By virtue of (3.4) , (3.5), and (2.4), equation (1.2) becomes

$$\phi_+'(x) + \frac{8i\phi_+(x)}{m\,b_+(x)} + \phi_-'(x) + \frac{8i\,\phi_-(x)}{m\,b_-(x)} = 4iV\alpha(x) + \frac{2ia\Gamma(a)}{\pi(a^2-x^2)} \tag{3.6}$$

Let us introduce the new function

$$F(\zeta) = \left[\phi'(\zeta)\,\frac{8i}{m}\,\frac{\phi(\zeta)}{b(\zeta)}\right]\sqrt{a^2-\zeta^2} \tag{3.7}$$

which is evidently holomorphic in T^* and continuously prolonged from the interior point of $[-a,a]$ to the sides of the upper and lower half-planes. At the points $-a$ and a , this function may have a singularity but only of order less than 1 [2]. Then (3.6) becomes

$$F_+(x) - F_-(x) = 4iV\alpha(x)\sqrt{a^2 - x^2} + \frac{2ia\Gamma(a)}{\pi\sqrt{a^2 - x^2}} \tag{3.8}$$

According to the Cauchy formula

$$F(\zeta) = \frac{1}{2\pi i}\int_{-a}^{a} \frac{F_+(t) - F_-(t)}{t - \zeta}\,dt + \frac{1}{2\pi i}\int_L \frac{F(t)}{t - \zeta}\,dt \tag{3.9}$$

where ζ is a point of T^*.

But by virtue of (3.7) and (3.8), (3.9) yields

$$F(\zeta) = \frac{2V}{\pi}\int_{-a}^{a} \frac{\alpha(t)\sqrt{a^2-t^2}}{t-\zeta}\,dt + \frac{a\Gamma(a)}{\pi^2}\int_{-a}^{a} \frac{1}{\sqrt{a^2 - t^2}}\,\frac{dt}{t - \zeta} \tag{3.10}$$

$$+ \frac{1}{2\pi i}\int_L \frac{\sqrt{a^2-t^2}\,\Phi'(t)}{t - \zeta}\,dt + \frac{4}{m\pi}\int_L \frac{\sqrt{a^2-t^2}}{b(t)}\,\frac{\Phi(t)}{t - \zeta}\,dt$$

Using the Cauchy formula and theorem, we easily obtain

$$\int_{-a}^{a} \frac{1}{\sqrt{a^2 - t^2}}\,\frac{dt}{t - \zeta} = \frac{\pi i}{\sqrt{a^2 - \zeta^2}}; \quad \frac{1}{2\pi i}\int_L \frac{\sqrt{a^2-t^2}\,\Phi'(t)}{t - \zeta}\,dt = 0 \tag{3.11}$$

Further, by virtue of (2.1) and (3.1), we have

$$\int_L \frac{\sqrt{a^2-t^2}}{b(t)}\,\frac{\Phi(t)}{t-\zeta}\,dt = \frac{1}{2\pi i}\int_{-a}^{a} \Gamma(\sigma)d\sigma \int_L \frac{p(t)\,dt}{(t-\zeta)(\sigma-t)}$$

$$= \int_{-a}^{a} \frac{\Gamma(\sigma)d\sigma}{\sigma - \zeta}\left[\frac{1}{2\pi i}\int_L \frac{p(t)}{t - \zeta}\,dt + \frac{1}{2\pi i}\int_L \frac{p(t)}{\sigma - t}\,dt\right]$$

$$= -\int_{-a}^{a} \frac{p(\sigma) - p(\zeta)}{\sigma - \zeta}\,\Gamma(\sigma)\,d\sigma \tag{3.12}$$

By virtue of (3.1) and (3.12), we obtain from (3.10)

$$F(\zeta) = \frac{2V}{\pi}\int_{-a}^{a} \frac{\alpha(t)\sqrt{a^2-t^2}}{t - \zeta}\,dt + \frac{ia\Gamma(a)}{\pi\sqrt{a^2-\zeta^2}} - \frac{4}{m\pi}\int_{-a}^{a} \frac{p(\sigma)-p(\zeta)}{\sigma - \zeta}\,\Gamma(\sigma)\,d\sigma \tag{3.13}$$

Hence by means of passing to the limit, we obtain the formula which we use later:

$$F_+(x) + F_-(x) = \frac{4V}{x}\int_{-a}^{a} \frac{\alpha(t)\sqrt{a^2-t^2}}{t - x}\, dt - \frac{g}{m\pi}\int_{-a}^{a} \frac{p(\sigma)-p(x)}{\sigma - x}\,\Gamma(\sigma)d\sigma \qquad (3.14)$$

From (3.7) we have

$$\Phi_+'(x) + \frac{gi}{mb(x)}\Phi_+(x) = \frac{F_+(x)}{\sqrt{a^2-x^2}}\ ;\ \Phi_-'(x) - \frac{gi}{mb(x)}\Phi_-(x) = \frac{-F_-(x)}{\sqrt{a^2-x^2}} \qquad (3.15)$$

Integrating these equations, we obtain

$$\Phi_+(x) = \Phi_+(0)e^{-i\theta(x)} + \int_0^x e^{i[\theta(t)-\theta(x)]}\frac{F_+(t)dt}{\sqrt{a^2-t^2}} \qquad (3.16)$$

$$\Phi_-(x) = \Phi_-(0)e^{i\theta(x)} - \int_0^x e^{-i[\theta(t)-\theta(x)]}\frac{F_-(t)\, dt}{\sqrt{a^2-t^2}}$$

where

$$\theta(x) = \frac{g}{m}\int_0^x \frac{dt}{b(t)} \qquad (-a\leq x\leq a) \qquad (3.17)$$

Because the function $\Gamma(x)$ is odd, from (3.1) we obtain

$$\Phi_+(0) = -\Phi_-(0) = \frac{1}{2}\Gamma(0)$$

Consequently, according to (3.4), we have from (3.16)

$$\Gamma(x) = \Gamma(0)\cos\theta(x) + \int_0^x \frac{\cos[\theta(t)-\theta(x)]}{\sqrt{a^2 - t^2}}\left[F_+(t) + F_-(t)\right] dt$$

$$+ \int_0^x \frac{\sin[\theta(t)-\theta(x)]\, i\left[F_+(t) - F_-(t)\right]}{\sqrt{a^2 - t^2}} dt \qquad (3.18)$$

On the basis of (3.8) and (3.14), this equation becomes

$$\Gamma(x) = \Gamma(0)\cos\theta(x) + \Gamma(a)\,\omega_1(x) + \int_{-a}^{x} K_0(x,\sigma)\,\Gamma(\sigma)d\sigma + g_0(x) \qquad (3.19)$$

where
$$\omega_0(x) = \frac{-2a}{\pi}\int_0^x \frac{\sin[\theta(t)-\theta(x)]}{\sqrt{a^2-t^2}}\,dt \tag{3.20}$$

$$g_0(x) = -4V\int_0^x \sin[\theta(t)-\theta(x)]\,\alpha(t)\,dt$$
$$+\frac{4V}{\pi}\int_0^x \frac{\cos[\theta(t)-\theta(x)]}{\sqrt{a^2-t^2}}\,dt\int_{-a}^{a}\frac{\alpha(\sigma)\sqrt{a^2-\sigma^2}}{\sigma-t}\,d\sigma \tag{3.21}$$

$$K_0(x,\sigma) = \frac{-8}{m\pi}\int_0^x \frac{\cos[\theta(t)-\theta(x)]}{\sqrt{a^2-t^2}}\;\frac{p(\sigma)-p(t)}{\sigma-t}\,dt \tag{3.22}$$

In case $\alpha =$ constant , as is easily seen,
$$g_0(x) = -4V\alpha\int_0^x\left\{\sin[\theta(t)-\theta(x)] + \frac{t[\cos\theta(t)-\theta(x)]}{\sqrt{a^2-t^2}}\right\}dt \tag{3.23}$$

Let us now make the assumption which usually applies in wing theory
$$\Gamma(a) = \Gamma(-a) = 0 \tag{3.24}$$

Let us consdder the two cases with the corresponding conditions
$$\cos\theta(a) \neq 0\ ; \quad \cos\theta(a) = 0 \tag{3.25}$$

For the first condition,,we obtain from (3.19) for $x = a$ by virtue of (3.24)
$$\Gamma(0) = -\frac{1}{\cos\theta(a)}\left\{\int_{-a}^{a} K_0(a,\sigma)\,\Gamma(\sigma)\,d\sigma + g_0(a)\right\} \tag{3.26}$$

Substituting this in (3/19) and taking into account (3.24) we obtain the Fredholm integral equation
$$\Gamma(x) - \int_{-a}^{a} K(x,\sigma)\,\Gamma(\sigma)\,d\sigma = g(x) \tag{3.27}$$

where
$$g(x) = g_0(x) - \frac{g_0(a)\cos\theta(x)}{\cos\theta(a)}\ ; \quad K(x,\sigma) = K_0(x,\sigma) - \frac{K_0(a,\sigma)\cos\theta(x)}{\cos\theta(a)} \tag{3.28}$$

By virtue of (3.24) with the second condition of (3.25), we obtain from (3.19)

$$\int_{-a}^{a} K_0(a,\sigma)\ \Gamma(\sigma)\ d\sigma + g_0(a) = 0 \tag{3.29}$$

Therefore, in this case we reduce to the system

$$\begin{aligned} \Gamma(x) &= \Gamma(0)\cos\theta(x) + \int_{-a}^{a} K_0(x,\sigma)\ \Gamma(\sigma)d\sigma + g_0(x) \\ &\int_{-a}^{a} K_0(a,\sigma)\ \Gamma(\sigma)d\sigma + g_0(a) = 0 \end{aligned} \tag{3.30}$$

The first of these equations contains the undetermined constant $\Gamma(0)$ which, generally speaking, must be determined from the second equation. If this constant is not determined from the specified equation, then it must be selected in such a way that the solution of the first equation of (3.30) will solve the Prandtl equation (1.2).

Equations (3.27) and (3.30) which are obtained, always have a simple structure since, in that important particular case when the function $p(x) = \sqrt{a^2-x^2}/b(x)$ is rational, the kernels of these equations degenerate; i.e. have the form

$$\sum_{i=1}^{n} \varphi_i(x)\psi_i(\sigma)$$

But in the latter case, the desired function $\Gamma(x)$ is evidently expressible in quadratures, since thesolution of equations (3.27) and (3.30) then reduces to the solution of a finite system of linear algebraic equations. The noted cases are of great practical interest because a wing of arbitrary form may always be approximated by any, previously assigned accurate, profiles of specified form. Consequently the method set down above practically always leads to the effective solution of Prandtl's equation.

Note: To derive integral equation (3.19), we assumed that $\sqrt{a^2-x^2}/b(x)$ is analytic on $[-a,a]$. However, this equation in its final form remains a Fredholm in the case when $\sqrt{a^2-x^2}/b(x)$ has, for example, a continuous first order derivative on $[-a,a]$. Equation (3.19), under these more general assumptions, was obtained by L.G.Magnaradze [4].

4. Let us consider some examples

Example 1. Let there be an elliptical wing

$$b(x) = b_0 \sqrt{1 - x^2/a^2} \tag{4.1}$$

In this case $p(x)$ = constant and according to (3.22) $K_0(x,\sigma) = 0$. Therefore, from (3.19), by virtue of (3.24), we have

$$\Gamma(x) = \Gamma(0)\cos\theta(x) + g_0(x) \tag{4.2}$$

while, as is evident from (3.20)

$$\theta(x) = \varkappa \sin^{-1} x/a \ ; \qquad \varkappa = 8a/mb_0 \tag{4.3}$$

The constant $\Gamma(0)$ is determined with the aid of (3.24) if the first condition of (3.25) occurs, which in the given case assumes the form $\cos \pi\varkappa/2 \neq 0$. If $\cos \pi\varkappa/2 = 0$ then the constant $\Gamma(0)$ must be determined from the condition that formula (4.2) represents the solution of the Prandtl equation. In the same way, for a wing of elliptic form at any angle of attack α, the solution of the Prandtl equation is constructed in evident form with the aid of (4.2). If α = constant, then by means of simple computations one is convinced that (4.2) assumes the form

$$\Gamma(x) = \Gamma(0)\cos\theta(x) - \frac{4a\alpha V}{1+\varkappa}\cos\theta(x) + \frac{4a\alpha V}{1+\varkappa}\sqrt{a^2-x^2}$$

Now satisfying the condition $\Gamma(a) = 0$ and assuming that $\cos \pi\varkappa/2 \neq 0$ we obtain the well-known formula (see [1] p. 203)

$$\Gamma(x) = \frac{4a\alpha V}{1+\varkappa}\sqrt{a^2 - x^2} \tag{4.4}$$

This formula remains valid in thecase when $\cos \pi\varkappa/2 = 0$. For this it is sufficient ~~that~~ that the function $\cos\theta(x) = \cos(\varkappa \arcsin x/a)$, where $\varkappa = 2k+1$ while k is an integer not satisfying the homogeneous Prandtl equation (α=0) ; this is easy to show with the aid of the substitution $x = a\cos\varphi$ if the well-known formula is used (see [1] ,p 202)

$$\int_0^\pi \frac{\cos n\varphi}{\cos\varphi - \cos\psi}\, d\varphi = \pi \frac{\sin n\psi}{\sin\psi} \qquad \text{(n-an integer)}$$

Example 2: Let us consider a wing of the form

$$b(x) = b_0\sqrt{1 - x^2/a^2}\left(\frac{1 + \nu x^2/a^2}{1 + \mu x^2/a^2}\right) \tag{4.5}$$

where μ and ν are each constants greater than -1.

Because in the given case

$$p(x) = \frac{a}{b_0}\,\frac{1 + \mu x^2/a^2}{1 + \nu x^2/a^2}$$

from (3.22) and the second relation of (3.28) by limiting the considerations to the case when $\cos\theta(a) \neq 0$, we find

$$K(x,t) = \frac{\varkappa(\nu - \mu)}{a^2 + \nu t^2}\varphi_1(x) + \frac{\varkappa(\nu - \mu)t}{a^2 + \nu t^2}\varphi_2(x) \tag{4.6}$$

where

$$\varphi_k(x) = \int_0^x \frac{\cos[\theta(\sigma) - \theta(x)]}{\sqrt{a^2 - \sigma^2}}\,\frac{\sigma^{k-1}}{1 + \nu\sigma^2/a^2}\, d\sigma$$
$$- \frac{\cos\theta(x)}{\cos\theta(a)}\int_0^a \frac{\cos[\theta(\sigma) - \theta(a)]}{\sqrt{a^2 - \sigma^2}}\,\frac{\sigma^{k-1}}{1 + \nu\sigma^2/a^2}\, d\sigma \quad (k=1,2) \tag{4.7}$$

while in the given case, as it is easy to find from (3.17) and (4.5)

$$\theta(x) = \frac{\varkappa\mu}{\nu}\arcsin x/a + \frac{\varkappa}{\nu}\frac{(\nu - \mu)}{\sqrt{1+\nu}}\arctan\frac{x\sqrt{1+\nu}}{\sqrt{a^2 - x^2}} \quad \text{for } \nu \neq 0 \tag{4.8}$$

$$\theta(x) = \varkappa\left(1 + \frac{\mu}{2}\right)\arcsin x/a - \frac{\varkappa\mu}{2}\, x\sqrt{a^2 - x^2} \quad \text{for } \nu = 0 \tag{4.9}$$

Substituting (4.6) in (3.27) and taking into account that $\Gamma(x)$ is even

$$\Gamma(x) = \varkappa(\nu - \mu)\varphi_1(x)\int_{-a}^a \frac{\Gamma(\sigma)\, d\sigma}{a^2 + \nu\sigma^2} + g(x) \tag{4.10}$$

Hence we find that

$$\Gamma(x) = g(x) + \left[\varkappa(\nu-\mu)\int_{-a}^{a}\frac{g(\sigma)\,d\sigma}{a^2+\nu\sigma^2}\right]\left[1-\varkappa(\nu-\mu)\int_{-a}^{a}\frac{\varphi_1(\sigma)\,d\sigma}{a^2+\nu\sigma^2}\right]^{-1}\varphi_1(x) \qquad (4.11)$$

where $g(x)$ is determined with the aid of the first formula of (3.28) and $\varphi_1(x)$ is obtained from (4.7) for $k = 1$. This formula yields the solution to the Prandtl equation in evident form for any angle of attack α in the case of wings given by (4.5). The importance of this formula is easy to estimate if it is noted that by varying the parameters μ and ν we may encompass by (4.5) a large number of practical important wing forms. For example, in case $\mu = 0$, $\nu = 0.9$, we obtain the wing which is almost rectangular as is evident from the following table

x/a	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
b(x)	1.00	1.02	1.03	1.05	1.06	1.06	1.03	0.95	0.75

$$b(x) = \sqrt{1 - x^2/a^2}\,(1 + 0.9x^2/a^2)$$

Let us note, finally, that following themethod outlined above, it is possible to obtain also the expression for $\Gamma(x)$ in the case of wing forms

$$b(x) = b_0\sqrt{1 - x^2/a^2}\left[\frac{1 + \nu_1 x^2/a^2 + \ldots + \nu_n x^{2n}/a^{2n}}{1 + \mu_1 x^2/a^2 + \ldots + \mu_n x^{2n}/a^{2n}}\right]$$

Wings of such form in the case when all $\mu_k = 0$ are considered by Schmidt [5]. However, his results appear less effective

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